

ON THE DISTINGUISHING NUMBER OF FUNCTIGRAPHS

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ABSTRACT. Let G_1 and G_2 be disjoint copies of a graph G , and let $g : V(G_1) \rightarrow V(G_2)$ be a function. A functigraph F_G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : g(u) = v\}$. In this paper, we extend the study of the distinguishing number of a graph to its functigraph. We discuss the behavior of the distinguishing number in passing from G to F_G and find its sharp lower and upper bounds. We also discuss the distinguishing number of functigraphs of complete graphs and join graphs.

1. PRELIMINARIES

Given a key ring of apparently identical keys to open different doors, how many colors are needed to identify them? This puzzle was given by Rubin [23] in 1980 for the first time. In this puzzle, there is no need for coloring to be proper. Indeed, one cannot find a reason why adjacent keys must be assigned different colors, whereas in other problems like storing chemicals, scheduling meetings a proper coloring is needed, and one with a small number of colors is required.

From the inspiration of this puzzle, Albertson and Collins [1] introduced the concept of the distinguishing number of a graph as follows: A labeling $f : V(G) \rightarrow \{1, 2, 3, \dots, t\}$ is called a *t-distinguishing* if no non-trivial automorphism of a graph G preserves the vertex labels. The *distinguishing number* of a graph G , denoted by $Dist(G)$, is the least integer t such that G has t -distinguishing labeling. For example, the distinguishing number of a complete graph K_n is n , the distinguishing number of a path P_n is 2 and the distinguishing number of a cycle C_n , $n \geq 6$ is 2. For a graph G of order n , $1 \leq Dist(G) \leq n$ [1]. If H is a subgraph of a graph G such that automorphism group of H is a subset of automorphism group of G , then $Dist(H) \leq Dist(G)$.

Harary [18] gave different methods (orienting some of the edges, coloring some of the vertices with one or more colors and same for the edges, labeling vertices or edges, adding or deleting vertices or edges) of destroying the symmetries of a graph. Collins and Trenk defined the distinguishing chromatic number in [13] where they used proper t -distinguishing for vertex labeling. They have also given a comparison between the distinguishing number, the distinguishing chromatic number and the

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chromatic number for families like complete graphs, paths, cycles, Petersen graph and trees etc. Kalinowski and Pilsniak [20] have defined similar graph parameters, the distinguishing index and the distinguishing chromatic index, they labeled edges instead of vertices. They have also given a comparison between the distinguishing number and the distinguishing index for a connected graph G of order $n \geq 3$. Boutin [7] introduced the concept of determining sets. In [4], Albertson and Boutin proved that a graph is t -distinguishable if and only if it has a determining set that is $(t-1)$ -distinguishable. They also proved that every Kneser graph $K_{n:k}$ with $n \geq 6$ and $k \geq 2$ is 2-distinguishable. A considerable literature has been developed in this area see [2, 3, 6, 8, 9, 11, 19, 22, 24].

Unless otherwise specified, all the graphs G considered in this paper are simple, non-trivial and connected. The *open neighborhood* of a vertex u of G is $N(u) = \{v \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of u is $N(u) \cup \{u\}$. Two vertices u, v are *adjacent twins* if $N[u] = N[v]$ and *non adjacent twins* if $N(u) = N(v)$. If u, v are adjacent or non adjacent twins, then u, v are *twins*. A set of vertices is called *twin-set* if every of its two vertices are twins. A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subset V(G)$ be any subset of vertices of G . The *induced subgraph*, denoted by $\langle S \rangle$, is the graph whose vertex set is S and whose edge set is the set of all those edges in $E(G)$ which have both end vertices in S .

The idea of permutation graph was introduced by Chartrand and Harary [10] for the first time. They defined the permutation graph as follows: a permutation graph consists of two identical disjoint copies of a graph G , say G_1 and G_2 , along with $|V(G)|$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation on $\{1, 2, \dots, |V(G)|\}$. Dorfler [14], introduced a mapping graph which consists of two disjoint identical copies of graph where the edges between the two vertex sets are specified by a function. The mapping graph was rediscovered and studied by Chen et al. [12], where it was called the functigraph. A functigraph is an extension of permutation graph. Formally the functigraph is defined as follows: Let G_1 and G_2 be disjoint copies of a connected graph G , and let $g : V(G_1) \rightarrow V(G_2)$ be a function. A *functigraph* F_G of a graph G consists of the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv : g(u) = v\}$. Linda et al. [15, 16] and Kang et al. [21] have studied the functigraph for some graph invariants like metric dimension, domination and zero forcing number. In [17], we have studied the fixing number of functigraph. The aim of this paper is to study the distinguishing number of functigraph.

Throughout the paper, we will denote the set of all automorphisms of a graph G by $\Gamma(G)$, the functigraph of G by F_G , $V(G_1) = A$, $V(G_2) = B$, $g : A \rightarrow B$ is a function, $g(V(G_1)) = I$, $|g(V(G_1))| = |I| = s$.

This paper is organized as follows. In Section 2, we give sharp lower and upper bounds for distinguishing number of functigraph. This section also establishes the

connections between the distinguishing number of graphs and their corresponding functigraphs in the form of realizable results. In Section 3, we provide the distinguishing number of functigraphs of complete graphs and join of path graphs. Some useful results related to these families have also been presented in this section.

2. BOUNDS AND SOME REALIZABLE RESULTS

The sharp lower and upper bounds on the distinguishing number of functigraphs are given in the following result.

Proposition 2.1. *Let G be a connected graph of order $n \geq 2$, then*

$$1 \leq \text{Dist}(F_G) \leq \text{Dist}(G) + 1.$$

Both bounds are sharp.

Proof. Obviously, $1 \leq \text{Dist}(F_G)$ by definition. Let $\text{Dist}(G) = t$ and f be a t -distinguishing labeling for graph G . Also, let $u_i \in A$ and $v_i \in B$, $1 \leq i \leq n$. We extend labeling f to F_G as: $f(u_i) = f(v_i)$ for all $1 \leq i \leq n$. We have following two cases for g :

- (1) If g is not bijective, then f as defined earlier is a t -distinguishing labeling for F_G . Hence, $\text{Dist}(F_G) \leq t$.
- (2) If g is bijective, then f as defined earlier destroys all non-trivial automorphisms of F_G except the flipping of G_1 and G_2 in F_G , for some choices of g . Thus, $\text{Dist}(F_G) \leq t + 1$.

For the sharpness of the lower bound, take $G = P_3$ and $g : A \rightarrow B$, be a function such that $g(u_i) = v_1, i = 1, 2$ and $g(u_3) = v_3$. For the sharpness of the upper bound, take G as rigid graph and g as identity function. \square

Since at least m colors are required to break all automorphisms of a twin set of cardinality m , so we have the following corollary.

Proposition 2.2. *Let U_1, U_2, \dots, U_t be disjoint twin sets in a connected graph G of order $n \geq 3$ and $m = \max\{|U_i| : 1 \leq i \leq t\}$,*

- (i) $\text{Dist}(G) \geq m$.
- (ii) *If $\text{Dist}(G) = m$, then $\text{Dist}(F_G) \leq m$.*

Lemma 2.3. *Let G be a connected graph of order $n \geq 2$ and g be a constant function, then $\text{Dist}(F_G) = \text{Dist}(G)$.*

Proof. Let $I = \{v\} \subset B$. Then $\Gamma(G) = \Gamma(< A \cup \{v\} >) \subset \Gamma(F_G)$. Thus, vertices in $A \cup \{v\}$ are labeled by $\text{Dist}(G)$ colors. Since g is a constant function, therefore all vertices in $V(F_G) \setminus \{A \cup \{v\}\}$ are not similar to any vertex in $A \cup \{v\}$ in functigraph F_G . Therefore, vertices in $V(F_G) \setminus \{A \cup \{v\}\}$ can also be labeled from these $\text{Dist}(G)$ colors. Hence, $\text{Dist}(F_G) = \text{Dist}(G)$. \square

Remark 2.4. Let G be a connected graph and $\text{Dist}(F_G) = m_1$ if g is constant and $\text{Dist}(F_G) = m_2$ if g is not constant, then $m_1 \geq m_2$.

A vertex v of degree at least three in a connected graph G is called a *major vertex*. Two paths rooted from the same major vertex and having the same length are called the *twin stems*.

We define a function $\psi : \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N} \setminus \{1\}$ as $\psi(m) = k$ where k is the least number such that $m \leq 2\binom{k}{2} + k$. For example, $\psi(19) = 5$. Note that ψ is well-defined.

Lemma 2.5. If a graph G has $t \geq 2$ twin stems of length 2 rooted at same major vertex, then $\text{Dist}(G) \geq \psi(t)$.

Proof. Let $x \in V(G)$ be a major vertex and $xu_iu'_i$ where $1 \leq i \leq t$ are twin stems of length 2 attach with x . Let $H = \langle \{x, u_i, u'_i\} \rangle$ and $k = \psi(t)$. We define a labeling $f : V(H) \rightarrow \{1, 2, \dots, k\}$ as:

$$\begin{aligned} f(x) &= k, \\ (1) \quad f(u_i) &= \begin{cases} 1 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k+1 \leq i \leq 2k \\ 3 & \text{if } 2k+1 \leq i \leq 3k \\ \vdots & \vdots \\ k & \text{if } (k-1)k+1 \leq i \leq k^2 \end{cases} \\ (2) \quad f(u'_i) &= \begin{cases} i \bmod(k) & \text{if } 1 \leq i \bmod(k) \leq k-1, \\ k & \text{if } i \bmod(k) = 0, \end{cases} \end{aligned}$$

Using this labeling, one can see that f is a t -distinguishing for H . Since permutations with repetition of k colors, when 2 of them are taken at a time is equal to $2\binom{k}{2} + k$, therefore at least k colors are needed to label the vertices in t -stems. Hence, k is the least integer for which G has k -distinguishing labeling. Since $\Gamma(H) \subseteq \Gamma(G)$, therefore $\text{Dist}(G) \geq \psi(t)$. \square

Lemma 2.6. For any integer $t \geq 2$, there exists a connected graph G and a function g such that $\text{Dist}(G) = t = \text{Dist}(F_G)$.

Proof. Construct the graph G as follows: let $P_{(t-1)^2+1} : x_1x_2x_3\dots x_{(t-1)^2+1}$ be a path. Join $(t-1)^2 + 1$ twin stems $x_1u_iu'_i$ where $1 \leq i \leq (t-1)^2 + 1$ each of length two with vertex x_1 of $P_{(t-1)^2+1}$. This completes construction of G . We first show that $\text{Dist}(G) = t$. For $t = 2$, we have two twin stems attach with x_1 , and hence $\text{Dist}(G) = 2$. For $t \geq 3$, we define a labeling $f : V(G) \rightarrow \{1, 2, 3, \dots, t\}$ as follows:

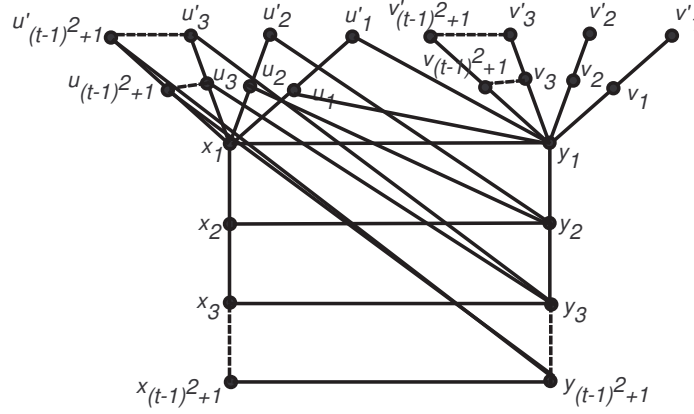


FIGURE 1. Graph with $\text{Dist}(G) = t = \text{Dist}(F_G)$.

$f(x_i) = t$, for all i , where $1 \leq i \leq (t-1)^2 + 1$.

$$f(u_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq t-1, \\ 2 & \text{if } t \leq i \leq 2(t-1), \\ 3 & \text{if } 2t-1 \leq i \leq 3(t-1), \\ \vdots & \vdots \\ t-1 & \text{if } (t-1)(t-2)+1 \leq i \leq (t-1)^2, \\ t & \text{if } i = (t-1)^2 + 1. \end{cases}$$

$$f(u'_i) = \begin{cases} i \bmod (t-1) & \text{if } 1 \leq i \bmod (t-1) \leq t-2 \text{ and } i \neq (t-1)^2 + 1, \\ t-1 & \text{if } i \bmod (t-1) = 0, \\ t & \text{if } i = (t-1)^2 + 1. \end{cases}$$

Using this labeling, one can see the unique automorphism preserving this labeling is the identity automorphism. Hence, f is a t -distinguishing. Since permutation with repetition of $t-1$ colors, when 2 of them are taken at a time is $2\binom{t-1}{2} + (t-1)$, therefore $(t-1)^2+1$ twin stems can be labeled by at least t -colors. Hence, t is the least integer such that G has t -distinguishing labeling. Now, we denote the corresponding vertices of G_2 as v_i, v'_i, y_i for all i , where $1 \leq i \leq (t-1)^2 + 1$ and construct a functigraph F_G by defining $g : V(G_1) \rightarrow V(G_2)$ as follows: $g(u_i) = g(u'_i) = y_i$, for all i , where $1 \leq i \leq (t-1)^2 + 1$ and $g(x_i) = g(y_i)$, for all i , where $1 \leq i \leq (t-1)^2 + 1$ as shown in the Figure 1. Thus, F_G has only symmetries of $(t-1)^2 + 1$ twin stems attach with y_1 . Hence, $\text{Dist}(F_G) = t$. \square

Consider an integer $t \geq 4$. We construct graph G similarly as in proof of Lemma 2.6 by taking a path $P_{(t-3)^2+1} : x_1 x_2 \dots x_{(t-3)^2+1}$ and attach $(t-3)^2 + 1$ twin stems $x_1 u_i u'_i$ where $1 \leq i \leq (t-3)^2 + 1$ with any one of its end vertex say x_1 . Using similar labeling and arguments as in proof of Lemma 2.6 one can see that f is $t-2$

distinguishing and $t-2$ is least integer such that G has $t-2$ distinguishing labeling. Define functigraph F_G , where $g : V(G_1) \rightarrow V(G_2)$ is defined by: $g(u_i) = g(u'_i) = y_i$, for all i , where $1 \leq i \leq (t-3)^2 + 1$, $g(x_i) = v_i$, for all i , where $1 \leq i \leq (t-3)^2 - 1$, $g(x_i) = y_i$, for all i , where $(t-3)^2 \leq i \leq (t-3)^2 + 1$. From this construction, F_G has only symmetries of 2 twin stems attach with y_1 , and hence $\text{Dist}(F_G) = 2$. Thus, we have the following result which shows that $\text{Dist}(G) + \text{Dist}(F_G)$ can be arbitrary large:

Lemma 2.7. *For any integer $t \geq 4$, there exists a connected graph G and a function g such that $\text{Dist}(G) + \text{Dist}(F_G) = t$.*

Consider $t \geq 3$. We construct graph G similarly as in proof of Lemma 2.6 by taking a path $P_{4(t-1)^2+1}$: $x_1x_2\dots x_{4(t-1)^2+1}$ and attach $4(t-1)^2+1$ twin stems $x_1u_iu'_i$, where $1 \leq i \leq 4(t-1)^2+1$ with x_1 . Using similar labeling and arguments as in proof of Lemma 2.6 one can see that f is $2t-1$ distinguishing and $2t-1$ is the least integer such that G has $2t-1$ distinguishing labeling. Let us now define g as $g(u_i) = g(u'_i) = y_i$, for all i , where $1 \leq i \leq 4(t-1)^2+1$, $g(x_i) = v_i$, for all i , where $1 \leq i \leq 3t^2-4t$ and $g(x_i) = y_i$, for all i , where $3t^2-4t+1 \leq i \leq 4(t-1)^2+1$. Thus, F_G has only symmetries of $(t-2)^2+1$ twin stems attach with y_1 , and hence $\text{Dist}(F_G) = t-1$. After making this type of construction, we have the following result which shows that $\text{Dist}(G) - \text{Dist}(F_G)$ can be arbitrary large:

Lemma 2.8. *For any integer $t \geq 3$, there exists a connected graph G and a function g such that $\text{Dist}(G) - \text{Dist}(F_G) = t$.*

3. THE DISTINGUISHING NUMBER OF FUNCTIGRAPHS OF SOME FAMILIES OF GRAPHS

In this section, we discuss the distinguishing number of functigraphs on complete graphs, edge deletion graphs of complete graph and join of path graphs.

Let G be the complete graph of order $n \geq 3$ and A and B be its two copies. We use following terminology for F_G in proof of Theorem 3.3: Let $I = \{v_1, v_2, \dots, v_s\}$ and $n_i = |\{u \in A : g(u) = v_i\}|$ for all i , where $1 \leq i \leq s$. Also, let $l = \max\{n_i : 1 \leq i \leq s\}$ and $m = |\{n_i : n_i = 1, 1 \leq i \leq s\}|$. From the definitions of l and m , we note that $2 \leq l \leq n-s+1$ and $0 \leq m \leq s-1$.

Using function $\psi(m)$ as defined in previous section, we have following lemma:

Lemma 3.1. *Let G be the complete graph of order $n \geq 3$ and g be a bijective function, then $\text{Dist}(F_G) = \psi(n)$.*

Proof. Let $A = \{u_1, u_2, \dots, u_n\}$ and $I = \{g(u_1), g(u_2), \dots, g(u_n)\} = B$. Also let $k = \psi(n)$. Let $f : V(F_G) \rightarrow \{1, 2, \dots, k\}$ be a labeling in which $f(u_i)$ is defined as in equation (1) and $f(g(u_i))$ as in equation (2) in proof of Lemma 2.5. Using this labeling one can see that f is a k -distinguishing labeling for F_G . Since permutation

with repetition of k colors, when 2 of them are taken at a time is equal to $2\binom{k}{2} + k$, therefore at least k colors are needed to label the vertices in F_G . Hence, k is the least integer for which F_G has k -distinguishing labeling. \square

Let G be a complete graph and let $g : A \rightarrow B$ be a function such that $2 \leq m \leq s$. Without loss of generality assume $u_1, u_2, \dots, u_m \in A$ are those vertices of A such that $g(u_i) \neq g(u_j)$ where $1 \leq i \neq j \leq m$ in B . Also $(u_i u_j)(g(u_i)g(u_j)) \in \Gamma(F_G)$ for all $i \neq j$ where $1 \leq i, j \leq m$. By using similar labeling f as defined in Lemma 3.1, at least $\psi(m)$ color are needed to break these automorphism in F_G . Thus, we have following proposition:

Proposition 3.2. *Let G be a complete graph of order $n \geq 3$ and g be a function such that $2 \leq m \leq s$, then $\text{Dist}(F_G) \geq \psi(m)$.*

The following result gives the distinguishing number of functigraphs of complete graphs.

Theorem 3.3. *Let $G = K_n$ be the complete graph of order $n \geq 3$, and let $1 < s \leq n - 1$, then*

$$\text{Dist}(F_G) \in \{n - s, n - s + 1, \psi(m)\}.$$

Proof. We discuss following cases for l :

- (1) If $l = n - s + 1 > 2$, then A contains $n - s + 1$ twin vertices and B contains $n - s$ twin vertices (except for $n = 3, 4$ where B contains no twin vertices). Also, there are $m (= s - 1)$ vertices in A which have distinct images in B . These m vertices and their distinct images are labeled by at least $\psi(m)$ colors (only 1 color if $m = 1$) by Proposition 3.2. Since $n - s + 1$ is the largest among $n - s + 1$, $n - s$ and $\psi(m)$. Thus, $n - s + 1$ is the least number such that F_G has $(n - s + 1)$ -distinguishing labeling. Thus, $\text{Dist}(F_G) = n - s + 1$.
- (2) If $l = n - s + 1 = 2$, then $\psi(m) \geq \max\{n - s + 1, n - s\}$, and hence $\text{Dist}(F_G) = \psi(m)$.
- (3) If $l < n - s$, then B contains largest set of $n - s$ twin vertices in F_G . Also, there are $m (\leq s - 2)$ vertices in A each of which has distinct image in B . Since $n - s \geq \psi(m)$, therefore $\text{Dist}(F_G) = n - s$.
- (4) If $l = n - s > 2$, then both A and B contain largest set of $n - s$ twin vertices in F_G . Also, there are $m (= s - 2)$ vertices in A which have distinct images in B . Since $n - s \geq \psi(m)$, therefore $\text{Dist}(F_G) = n - s$.
- (5) If $l = n - s = 2$, then we take two subcases:
 - (a) If $1 < s \leq \lfloor \frac{n}{2} \rfloor + 1$, then both A and B contain largest set of $n - s$ twin vertices in F_G . Also, there are $m (= s - 2)$ vertices in A which have distinct images in B . Since $n - s \geq \psi(m)$ (if $\psi(m)$ exists), therefore $\text{Dist}(F_G) = n - s$.
 - (b) If $\lfloor \frac{n}{2} \rfloor + 1 < s \leq n - 1$, then $\psi(m) \geq \max\{n - s + 1, n - s\}$, and hence $\text{Dist}(F_G) = \psi(m)$.

□

Let e^* be an edge of a connected graph G . Let $G - ie^*$ be the graph obtained by deleting i edges from graph G . A vertex v of a graph G is called *saturated* if it is adjacent to all other vertices of G .

We define a function $\phi : \mathbb{N} \rightarrow \mathbb{N} \setminus \{1\}$ as $\phi(i) = k$, where k is the least number such that $i \leq \binom{k}{2}$. For instance, $\phi(32) = 9$. Note that ϕ is well defined.

Theorem 3.4. *Let G be the complete graph of order $n \geq 5$ and $G_i = G - ie^*$ for all i where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and e^* joins two saturated vertices of the graph G . If g is a constant function, then*

$$Dist(F_{G_i}) = \max\{n - 2i, \phi(i)\}.$$

Proof. On deleting i edges e^* from G , we have $n - 2i$ saturated vertices and i twin sets each of cardinality two. We will now show that exactly $\phi(i)$ colors are required to label vertices of all i twin sets. We observe that, a vertex in a twin set can be mapped on any one vertex in any other twin set. Since two vertices in a twin set are labeled by a unique pair of colors out of $\binom{k}{2}$ pairs of k colors, therefore at least k colors are required to label vertices of i twin sets. Now, we discuss the following two cases for $\phi(i)$:

- (1) If $\phi(i) \leq n - 2i$, then number of colors required to label $n - 2i$ saturated vertices is greater than or equal to number of colors required to label vertices of i twin sets. Thus, we label $n - 2i$ saturated vertices with exactly $n - 2i$ colors and out of these $n - 2i$ colors, $\phi(i)$ colors will be used to label vertices of i twin sets.
- (2) If $\phi(i) > n - 2i$, then number of colors required to label $n - 2i$ saturated vertices is less than the number of colors required to label vertices of i twin sets. Thus, we label vertices of i twin sets with $\phi(i)$ colors and out of these $\phi(i)$ colors, $n - 2i$ colors will be used to label saturated vertices in G_i .

If g is constant, then by using same arguments as in the proof of Lemma 2.3, $Dist(F_{G_i}) = Dist(G_i)$. □

Suppose that $G = (V_1, E_1)$ and $G^* = (V_2, E_2)$ be two graphs with disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2 . The *join* of G and G^* is the graph $G + G^*$, in which $V(G + G^*) = V_1 \cup V_2$ and $E(G + G^*) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Theorem 3.5. [5] *Let G and G^* be two connected graphs, then $Dist(G + G^*) \geq \max\{Dist(G), Dist(G^*)\}$.*

Proposition 3.6. *Let P_n be a path graph of order $n \geq 2$, then for all $m, n \geq 2$ and $1 < s < m + n$, $1 \leq Dist(F_{P_m + P_n}) \leq 3$.*

Proof. Let $P_m : v_1, \dots, v_m$ and $P_n : u_1, \dots, u_n$. We discuss following cases for m, n .

- (1) If $m = 2$ and $n = 2$, then $P_2 + P_2 = K_4$, and hence $1 \leq \text{Dist}(F_{K_4}) \leq 3$ by Theorem 3.3.
- (2) If $m = 2$ and $n = 3$, then $P_2 + P_3$ has 3 saturated vertices. Thus, $1 \leq \text{Dist}(F_{P_2+P_3}) \leq 4$ by Proposition 2.1. However, for all s where $2 \leq s \leq 4$ and all possible definitions of g in $F_{P_2+P_3}$, one can see $1 \leq \text{Dist}(F_{P_2+P_3}) \leq 3$.
- (3) If $m = 3$ and $n = 3$, then a labeling $f : V(P_3 + P_3) \rightarrow \{1, 2, 3\}$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x = v_1, v_2 \\ 2 & \text{if } x = v_3, u_3 \\ 3 & \text{if } x = u_1, u_2 \end{cases}$$

is a distinguishing labeling for $P_3 + P_3$, and hence $\text{Dist}(P_3 + P_3) = 3$. Thus, $1 \leq \text{Dist}(F_{P_3+P_3}) \leq 4$ by Proposition 2.1. However, for all s where $2 \leq s \leq 5$ and all possible definitions of g in $F_{P_3+P_3}$, one can see $1 \leq \text{Dist}(F_{P_3+P_3}) \leq 3$.

- (4) If $m \geq 2$ and $n \geq 4$, then a labeling $f : V(P_m + P_n) \rightarrow \{1, 2\}$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x = v_1, u_2, \dots, u_n \\ 2 & \text{if } x = u_1, v_2, \dots, v_m \end{cases}$$

is a distinguishing labeling for $P_m + P_n$, and hence $\text{Dist}(P_m + P_n) = 2$. Thus, result follows by Proposition 2.1.

□

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